ON THE DEVELOPMENT OF A MAGNETOHYDRODYNAMIC BOUNDARY LAYER

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The problem of the development of a plane magnetohydrodynamic boundary layer in a viscous incompressible fluid is investigated for a time dependence of the velocity of the outer boundary of the boundary layer and of the magnetic field in the form (1). A solution of the problem is found under the assumption that the body motion starts impulsively and continues at equal acceleration.

A solution of the problem of magnetohydrodynamic boundary layer development on a body which starts to move uniformly, at equal acceleration, or with acceleration, in a fluid at rest, has been obtained in a number of papers [1-8]. An idea of S. G. Slavchev [9] is used below to investigate magnetohydrodynamic boundary layer development in a cylinder. Let us consider the following growth law for the velocity on the outer boundary of the boundary layer and of the magnetic field with time:

$$U(x,\ t) = V(x)\,\Omega(t),\ B(x,\ t) = B_0(x)\sqrt{\Omega(t)},$$
$$\Omega(t) = At^{\alpha}(1 + A_1 t^n). \tag{1}$$

Let us note that an analogous problem was considered for $A_1 = 0$ in [4].

Boundary Layer Equations. Let us examine the plane nonstationary flow of a viscous incompressible fluid with an applied magnetic field normal to the body surface. No electrical field is imposed and the electrical field intensity vector E is taken zero everywhere [10]. Then the boundary equations in the coordinate system coupled to the body are [4]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}(u - U);\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

let us integrate them with the following initial and boundary conditions

$$\begin{array}{lll} u = U(x,\ t),\ v = 0 & \text{for}\ \ y = 0,\ t = 0; & \\ u = 0,\ v = 0 & \text{for}\ \ y = 0 & \multirow{2}{*}{$\Big\}\ t > 0.$} \\ u \to U(x,\ t) & \text{for}\ \ y \to \infty & \end{array} \tag{3}$$

If we introduce a new variable in the dimensionless form $\eta = y/2\sqrt{\nu t}$ instead of the coordinate y measured along the normal to the body contour, and also the stream function $\psi(x, y, t) = 2\sqrt{\nu t}\,\varphi(x, \eta, t)$, we then obtain the following equation for the function $\varphi(x, \eta, t)$:

$$\frac{1}{4}\cdot\frac{\partial^3\varphi}{\partial\eta^3} + \frac{1}{2}\eta\frac{\partial^2\varphi}{\partial\eta^2} + \zeta(t)\left(1 - \frac{\partial\varphi}{\partial\eta}\right) - t\frac{\partial^2\varphi}{\partial t\partial\eta}$$
$$+ \frac{dV}{dx}\Omega t\left[1 + \varphi\frac{\partial^2\varphi}{\partial\eta^2} - \left(\frac{\partial\varphi}{\partial\eta}\right)^2\right] + V\Omega t\left(\frac{\partial\varphi}{\partial x}\cdot\frac{\partial^2\varphi}{\partial\eta^2} - \frac{\partial\varphi}{\partial\eta}\cdot\frac{\partial^2\varphi}{\partial x\partial\eta}\right) + N\frac{dV}{dx}\Omega t\left(1 - \frac{\partial\varphi}{\partial\eta}\right) = 0 \tag{4}$$

with the boundary conditions

$$\varphi = \frac{\partial\varphi}{\partial\eta} = 0\ \ \text{for}\ \ \eta = 0;\quad \frac{\partial\varphi}{\partial\eta} \to 1\ \ \text{for}\ \ \eta \to \infty. \tag{5}$$

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Hence

$$\zeta(t)=\frac{\dot{\Omega}t}{\Omega}=\alpha+\frac{nA_1t^n}{1+A_1t^n};\quad N=\sigma B_0^2/\rho\,\frac{dV}{dx},$$

where the dot denotes differentiation with respect to time. We henceforth assume $\zeta \neq \alpha$ and $A_1 > 0$.

Solutions of Equation (4). Following the method of S. G. Slavchev, let us introduce the infinite system of independent variables

$$p_k=V^{k-1}\frac{d^kV}{dx^k}\,\Omega^kt^k,\; k=1,\,2,\,\ldots,$$

satisfying the recursion dependences

$$V\Omega t\frac{\partial p_k}{\partial x}=p_{k+1}+(k-1)\,p_1p_k,\quad t\frac{\partial p_k}{\partial t}=(1+\zeta)\,kp_k,$$

and let us make a change of variables by means of the formulas

$$\frac{\partial}{\partial\eta}=\frac{\partial}{\partial\eta},\quad t\frac{\partial}{\partial t}=t\frac{\partial}{\partial t}+(1+\zeta)\sum_{k=1}^{\infty}kp_k\frac{\partial}{\partial p_k},$$

$$V\Omega t\frac{\partial}{\partial x}=\sum_{k=1}^{\infty}\left[p_{k+1}+(k-1)\,p_1p_k\right]\frac{\partial}{\partial p_k}.$$

We consequently obtain the following equation

$$\frac{1}{4}\cdot\frac{\partial^3\varphi}{\partial\eta^3}+\frac{1}{2}\eta\frac{\partial^2\varphi}{\partial\eta^2}+\zeta\left(1-\frac{\partial\varphi}{\partial\eta}\right)-(1+\zeta)\sum_{k=1}^{\infty}kp_k\frac{\partial^2\varphi}{\partial\eta\partial p_k}-t\frac{\partial^2\varphi}{\partial t\partial\eta}+p_1\left[1+\varphi\frac{\partial^2\varphi}{\partial\eta^2}-\left(\frac{\partial\varphi}{\partial\eta}\right)^2\right]$$

$$+\sum_{k=1}^{\infty}\left[p_{k+1}+(k-1)p_1p_k\right]\left(\frac{\partial\varphi}{\partial p_k}\cdot\frac{\partial^2\varphi}{\partial\eta^2}-\frac{\partial\varphi}{\partial\eta}\cdot\frac{\partial^2\varphi}{\partial\eta\partial p_k}\right)+Np_1\left(1-\frac{\partial\varphi}{\partial\eta}\right)=0. \tag{6}$$

Substituting a power series expansion of the function φ

$$\varphi=\varphi_0(\eta,\,t)+\varphi_1(\eta,\,t)\,p_1+\varphi_{11}(\eta,\,t)\,p_1^2+\ldots \tag{7}$$

into this equation and collecting terms with identical combinations of the variables p_k, we obtain the following system of equations for the functions φ_0, φ_1, etc.:

$$\frac{1}{4}\cdot\frac{\partial^3\varphi_0}{\partial\eta^3}+\frac{1}{2}\eta\frac{\partial^2\varphi_0}{\partial\eta^2}+\zeta\left(1-\frac{\partial\varphi_0}{\partial\eta}\right)-t\frac{\partial^2\varphi_0}{\partial t\partial\eta}=0,$$

$$\frac{1}{4}\cdot\frac{\partial^3\varphi_1}{\partial\eta^3}+\frac{1}{2}\eta\frac{\partial^2\varphi_1}{\partial\eta^2}-(1+2\zeta)\frac{\partial\varphi_1}{\partial\eta}-t\frac{\partial^2\varphi_1}{\partial t\partial\eta}=-\left[1+\varphi_0\frac{\partial^2\varphi_0}{\partial\eta^2}-\left(\frac{\partial\varphi_0}{\partial\eta}\right)^2\right]-N\left(1-\frac{\partial\varphi_0}{\partial\eta}\right) \tag{8}$$

with the conditions

$$\varphi_0=\frac{\partial\varphi_0}{\partial\eta}=0\quad\text{for }\eta=0;\quad\frac{\partial\varphi_0}{\partial\eta}\to1\quad\text{for }\eta\to\infty,$$

$$\varphi_1=\frac{\partial\varphi_1}{\partial\eta}=0\quad\text{for }\eta=0:\quad\frac{\partial\varphi_1}{\partial\eta}\to0\quad\text{for }\eta\to\infty. \tag{9}$$

Analyzing the system (8) we arrive at the conclusion that this solution should be sought in the form

$$\varphi_0(\eta,\,t)=f_0(\eta)+(\zeta-\alpha)\,f_n(\eta)=f_0(\eta)+\frac{nA_1t^n}{1+A_1t^n}f_n(\eta),$$

$$\varphi_1(\eta,\,t)=g_0(\eta)+(\zeta-\alpha)\,g_n^{(1)}(\eta)+(\zeta-\alpha)^2\,g_n^{(2)}(\eta), \tag{10}$$

where it turns out, after substituting (10) into (8), that the functions $f_0(\eta)$ etc., are determined by the differential equations:

$$f_0'''+2\eta f_0''-4\alpha f_0'=-4\alpha,$$

$$f_n'''+2\eta f_n''-4(\alpha+n)\,f_n'=4\,(f_0'-1),$$

$$
\begin{aligned}
&g_0''' + 2\eta g_0'' - 4(2\alpha+1)g_0' = -4(1 + f_0 f_0'' - f_0'^2) - 4N(1 - f_0'),\\
&g_n^{(1)'''} + 2\eta g_n^{(1)''} - 4(2\alpha+n+1)g_n^{(1)'}\\
&\quad = 4(2g_0' - f_0 f_n'' - f_n f_0'' + 2f_0' f_n') + 4Nf_n',\\
&g_n^{(2)'''} + 2\eta g_n^{(2)''} - 4(2\alpha+2n+1)g_n^{(2)'} = 4(g_n^{(1)'} - f_n f_n'' + f_n'^2),
\end{aligned}
\tag{11}
$$

where the prime denotes differentiation with respect to η.

A relationship which the function

$$
t\zeta + \zeta(\zeta - \alpha) = (n+\alpha)(\zeta - \alpha)
$$

satisfies, is hence used.

Henceforth, let us examine the case $\alpha = 0$ and $n = 1$. This means that the body motion starts impulsively at a velocity A and continues at an equal acceleration with acceleration AA_1. The problem then reduces to solving the following ordinary differential equations:

$$
\begin{gathered}
f_0''' + 2\eta f_0'' = 0, \quad f_1''' + 2\eta f_1'' - 4f_1' = 4(f_0' - 1),\\
f_0(0) = f_0'(0) = f_1(0) = f_1'(0) = 0, \quad f_0'(\infty) \to 1, \quad f_1'(\infty) \to 0,\\
g_0''' + 2\eta g_0'' - 4g_0' = -4(1 + f_0 f_0'' - f_0'^2) - 4N(1 - f_0'),\\
g_1^{(1)'''} + 2\eta g_1^{(1)''} - 8g_1^{(1)'} = 4(2g_0' - f_0 f_1'' - f_1 f_0'' + 2f_0' f_1') + 4Nf_1',\\
g_1^{(2)'''} + 2\eta g_1^{(2)''} - 12 g_1^{(2)'} = 4(g_1^{(1)'} - f_1 f_1'' + f_1'^2),\\
g_0(0) = g_0'(0) = g_1^{(i)}(0) = g_1^{(i)'}(0) = 0; \quad g_0'(\infty) \to 0, \quad g_1^{(i)'}(\infty) \to 0.
\end{gathered}
\tag{12}
$$

We do not give here a detailed exposition of the method of solving the system of equations (12) obtained, but only recall that the solution of the system can be obtained in closed form. This has been elucidated in greater detail in [11-13]. Therefore, the solution of the system (12) satisfying the appropriate boundary is:

$$
f_0(\eta) = \eta \operatorname{erf}\eta + \frac{1}{\sqrt{\pi}}(e^{-\eta^2} - 1),
$$

$$
f_1(\eta) = \frac{2}{3}\eta^3 \operatorname{erf}\eta + \frac{1}{3\sqrt{\pi}}(2\eta^2 - 1)e^{-\eta^2} - \frac{2}{3}\eta^3 + \frac{1}{3\sqrt{\pi}},
$$

$$
\begin{aligned}
g_0'(\eta) = &-\left(1 + N + \frac{2}{3\pi}\right)(1 + 2\eta^2) + \left(\frac{1}{2} + N + \frac{2}{3\pi}\right)\\
&\times\left[(1 + 2\eta^2)\operatorname{erf}\eta + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right] + \left(\eta^2 - \frac{1}{2}\right)\operatorname{erf}^2\eta + \left(\frac{3}{\sqrt{\pi}}\eta e^{-\eta^2} - N\right)\operatorname{erf}\eta + \frac{2}{\pi}e^{-2\eta^2} - \frac{4}{3\pi}e^{-\eta^2} + N + 1,
\end{aligned}
$$

$$
\begin{aligned}
g_1^{(1)'}(\eta) = &-\left(\frac{4}{3} + 2N + \frac{32}{45\sqrt{\pi}}\right)\left(\frac{3}{4} + 3\eta^2 + \eta^4\right) + \left(1 + 2N - \frac{32}{45\sqrt{\pi}}\right)\left[\left(\frac{3}{4} + 3\eta^2 + \eta^4\right)\operatorname{erf}\eta\right.\\
&\left.+ \frac{1}{\sqrt{\pi}}\left(\eta^3 + \frac{5}{2}\eta\right)e^{-\eta^2}\right] + \left(\frac{1}{4} - \eta^2 + \frac{1}{4}\eta^4\right)\operatorname{erf}^2\eta + \left[\frac{1}{\sqrt{\pi}}\left(\eta^3 - \frac{13}{6}\eta\right)e^{-\eta^2} - 2\left(1 + 3N + \frac{4}{3\pi}\right)\eta^2\right.\\
&\left.- \frac{8}{3\sqrt{\pi}}\eta - 1 - \frac{3}{2}N - \frac{4}{3\pi}\right]\operatorname{erf}\eta + \frac{2}{3\pi}(\eta^2 - 2)e^{-2\eta^2} - \frac{1}{3\sqrt{\pi}}\left[\frac{8}{5} + \left(\frac{21}{2} + 18N + \frac{8}{\pi}\right)\eta + \eta^3\right]e^{-\eta^2}\\
&+ 2\left(2 + 3N + \frac{4}{3\pi}\right)\eta^2 + \frac{8}{3\sqrt{\pi}}\eta + 1 + \frac{3}{2}N + \frac{4}{3\pi},
\end{aligned}
$$

$$
\begin{aligned}
g_1^{(2)'}(\eta) = &\left[\frac{8}{21\sqrt{\pi}}\left(1 - \frac{8}{15\sqrt{\pi}}\right) - \frac{1}{2}N\right]\left(1 + 6\eta^2 + 4\eta^4 + \frac{8}{15}\eta^6\right)\\
&+ \left[\frac{1}{2}\left(N - \frac{5}{6}\right) - \frac{8}{21\sqrt{\pi}}\left(1 - \frac{8}{15\sqrt{\pi}}\right)\right]\left[\left(1 + 6\eta^2 + 4\eta^4 + \frac{8}{15}\eta^6\right)\operatorname{erf}\eta\right.\\
&\left.+ \frac{8}{15\sqrt{\pi}}\left(\eta^5 + 7\eta^3 + \frac{33}{4}\eta\right)e^{-\eta^2}\right] + \left(\frac{1}{2}\eta^2 + \frac{2}{9}\eta^6\right)\operatorname{erf}^2\eta\\
&+ \left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\eta - \frac{5}{9}\eta^3 + \frac{4}{9}\eta^5\right)e^{-\eta^2} + \frac{1}{3}\left(5 - 6N + \frac{32}{15\sqrt{\pi}}\right)\eta^4\right.\\
&+ \left(2 - 3N + \frac{4}{3\pi} + \frac{32}{15\sqrt{\pi}}\right)\eta^2 + \frac{8}{5\sqrt{\pi}}\eta
\end{aligned}
$$

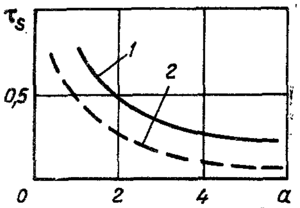

Fig. 1. Dependence of the dimensionless time of separation on the parameters a and N: 1) N = 0, 2) N = 0.5.

$$+\frac{1}{3}\left(\frac{5}{4}-\frac{3}{2}N+\frac{2}{\pi}+\frac{8}{5\sqrt{\pi}}\right)\Bigg]\operatorname{erf}\eta+\frac{1}{9\pi}\left(2-5\eta^2+2\eta^4\right)e^{-2\eta^2}$$
$$+\frac{2}{3\sqrt{\pi}}\left[\frac{2}{35}\left(4+\frac{17}{\sqrt{\pi}}\right)+\left(\frac{7}{2}-3N+\frac{2}{\pi}+\frac{8}{3\sqrt{\pi}}\right)\eta\right.$$
$$\left.+\left(3-3N+\frac{16}{15\sqrt{\pi}}\right)\eta^3\right]e^{-\eta^2}+\frac{2}{3}\left(3N-\frac{16}{15\sqrt{\pi}}\right)\eta^4$$
$$+\frac{1}{3}\left(9N-\frac{4}{\pi}-\frac{32}{5\sqrt{\pi}}\right)\eta^2-\frac{8}{5\sqrt{\pi}}\eta+\frac{1}{3}\left(\frac{3}{2}N-\frac{2}{\pi}-\frac{16}{5\sqrt{\pi}}\right).$$

The equations for higher order approximations can also be reduced to the integration of ordinary differential equations which can be performed in quadratures.

The time between the beginning of body motion and the time of boundary layer separation $\tau_S = A_1 t$ (if separation occurs) is found from the equation

$$\frac{2\sqrt{\nu t}}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}=f_0''(0)+\frac{\tau_s}{1+\tau_s}f_1''(0)+\tau_s\frac{dV}{dx}\frac{A}{A_1}\left[(1+\tau_s)g_0''(0)+\tau_s g_1^{(1)''}(0)+\frac{\tau_s^2}{1+\tau_s}g_1^{(2)''}(0)\right]=0, \quad (13)$$

where

$$f_0''(0)=f_1''(0)=\frac{2}{\sqrt{\pi}},\quad g_0''(0)=\frac{2}{\sqrt{\pi}}(1.424+N),$$
$$g_1^{(1)''}(0)=-\frac{2}{\sqrt{\pi}}(1.069+0.5N),\quad g_1^{(2)''}(0)=\frac{2}{\sqrt{\pi}}(0.195+0.1N). \quad (14)$$

Using (13), we can determine the time of boundary layer separation of any body, particularly a cylinder. For a cylinder V(x) = 2sin x/R, from which it follows that separation first sets in near the rear stagnation point for which the derivative of the function V(x) equals (−2/R). Substituting the values (14) and introducing the parameter $a = A/RA_1$ into (13), we obtain a formula for the time interval τ_S between the beginning of the motion and the time of origination of boundary layer separation

$$1+\frac{\tau_s}{1+\tau_s}-2a\tau_s\left[(1.424+N)(1+\tau_s)-(1.069+0.5N)\tau_s+(0.195+0.1N)\frac{\tau_s^2}{1+\tau_s}\right]=0.$$

Results of computing the dimensionless time τ_S for boundary layer separation on the contour of a cylinder are represented in the figure for diverse values of the magnetic parameter N.

It is seen from Fig. 1 how rapidly separation occurs for N = 0 and 0.5, i.e., how soon after insertion of the magnetic field.

NOTATION

x and y	are the coordinates measured along the contour and along the normal to the body surface, respectively;
t	is the time;
$\tau_S = A_1 t$	is the dimensionless time of separation;
u and v	are the velocity components along x and y;
U(x, t)	is the velocity on the outer boundary of the boundary layer;
$B_0(x)$	is the magnetic induction;
ρ	is the density;
ν	is the viscosity;
σ	is the electrical conductivity of the fluid;
R	is the cylinder radius;
A and A_1	are the non-negative constants;
α	is any non-negative number;
n	is an integer;
$a = A/RA_1$	is a parameter;
N	is a parameter with the meaning of the Stuart number.

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